PLASTIC PROPERTIES OF MATRIX COMPOSITES IN BENDING

V. V. Novikov and O. B. Papkovskaya UDC 518.12:536.19

Using the methods of integrated cross-sections and elastic solutions, we solve an elastico-plastic problem of bending of a Kirchhoff inhomogeneous square plate. The elastico-plastic properties and the effective yield stress of the inhomogeneous plate are calculated on an electronic computer. The computational results form the basis for a qualitative analysis and for the conclusions made.

Introduction. The first attempts at a theory of plasticity were undertaken in the early 1870s. The basic hypotheses and differential equations of so-called flow theory are given in the works of Saint-Venant, Levi, and Mises. The theory of plasticity is being developed today in connection with the appearance of new materials, and composite ones in particular. The study of their plastic properties is important because they include the majority of building materials. The problem is being solved by various methods, such as the method of R-functions [1], of elastic solutions [2], of expansion in the loading parameter [3], and others.

Below the plastic properties of matrix composites in bending are investigated using the methods of integrated cross-sections and elastic solutions.

Statement of the Problem. We consider the elastico-plastic problem of bending of an inhomogeneous square plate of constant thickness h ($0 \le x, y \le a$) consisting of two different materials, in two corresponding regions G_1 and G_2. To state and solve the problem we use a mathematical model of bending of a Kirchhoff plate.

The symmetry center of the plate is $x = a/2$, $y = a/2$. The materials that comprise the inner G_1 (inclusion) and outer G_2 (matrix) regions of the plate have yield stresses σ_{y1} and σ_{y2} and flexural rigidities D_1 and D_2, respectively. In this case $a/2 - \varepsilon \le x, y \le a/2 + \varepsilon$, with $0 < \varepsilon < a/2$.

For the sake of definiteness of the problem, we will assume that the lateral faces of the plate are clamped in a rigid base. The upper face of the plate is subjected to a uniformly distributed load with intensity g = const acting along the normal to the face.

The initial differential equation of the bending of the plate (the S. Germain equation) is

$$Lw = \frac{\partial^4 w(x, y)}{\partial x^4} + 2\frac{\partial^4 w(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y)}{\partial y^4} = \begin{cases} \dfrac{g}{D_1}, & \text{if } (x, y) \subset G_1, \\ \dfrac{g}{D_2}, & \text{if } (x, y) \subset G_2, \end{cases} \tag{1}$$

where

$$D_i = E_i h^3 / (12(1 - \nu_i^2)), \quad i = 1, 2; \tag{2}$$

w is the sagging of the plate along the z axis.

At the phase interface $B = G_1 \cap G_2$ the condition of continuity of displacements is satisfied.

The boundary conditions are written as follows:

$$w|_{x=0,a} = \left.\frac{\partial w(x, y)}{\partial x}\right|_{x=0,a} = 0; \quad w|_{y=0,a} = \left.\frac{\partial w(x, y)}{\partial y}\right|_{y=0,a} = 0. \tag{3}$$

Odessa State Polytechnical University, Ukraine. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 70, No. 6, pp. 1045-1048, 1997. Original article submitted July 9, 1996.

The objective of the present work is to find the effective yield stress σ_{eff} of the composite considered for various geometrical parameters of inclusion by means of the method of integral cross sections.

Plasticity. The transition of the material into the state under consideration is determined by the following condition [4]:

$$f(\sigma_1, \sigma_2, \sigma_3) = 0 \, ,$$

where σ_1, σ_2, and σ_3 are the principal normal stresses. In the system of coordinates σ_1, σ_2, and σ_3 this function describes the yield surface.

There are several criteria of the plasticity condition. For example, the Huber–Mises criterion of plasticity determines the transition from an elastic to a plastic state in the vicinity of a point on condition that

$$\sigma_i = \sigma_y \, , \tag{4}$$

where σ_y is the yield stress of the material; σ_i is the intensity of stresses.

Owing to porosity, the Huber–Mises condition is not valid for powder materials by virtue of both genetic and acquired porosity.

According to the Huber–Mises criterion of plasticity (4), the equation of the yield surface can be written as follows [4]:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 2\sigma_y^2 = 0 \, . \tag{5}$$

In the components of the stress tensor Eq. (5) can be rewitten as

$$(\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + (\sigma_{11} - \sigma_{22})^2 + 6\,(\sigma_{23}^2 - \sigma_{31}^2 + \sigma_{12}^2) = 2\sigma_y^2 \, . \tag{6}$$

Effective Yield Stress. We obtain the Huber–Mises condition of the plasticity criterion for the bending of plates from Eq. (6) at $\sigma_3 = 0$:

$$(\sigma_{11} - \sigma_{22})^2 + \sigma_{22}^2 + \sigma_{11}^2 + 6\sigma_{12}^2 = 2\sigma_y^2 \, . \tag{7}$$

We will assume that condition (7) is valid for both separate components (matrices and inclusions) and the entire composite material, which is characterized by the effective yield stress $\sigma_{y.ef}$, i.e.,

$$\frac{1}{2}\left[\left\langle\left(\frac{\sigma_{11}}{\sigma_y} - \frac{\sigma_{22}}{\sigma_y}\right)^2 + \left(\frac{\sigma_{22}}{\sigma_y}\right)^2 + \left(\frac{\sigma_{11}}{\sigma_y}\right)^2 + 6\left(\frac{\sigma_{12}}{\sigma_y}\right)^2\right\rangle\right] =$$

$$= \frac{1}{2}\left[\left(\frac{\langle\sigma_{11}\rangle}{\sigma_{y.ef}} - \frac{\langle\sigma_{22}\rangle}{\sigma_{y.ef}}\right)^2 + \left(\frac{\langle\sigma_{22}\rangle}{\sigma_{y.ef}}\right)^2 + \left(\frac{\langle\sigma_{11}\rangle}{\sigma_{y.ef}}\right)^2 + 6\left(\frac{\langle\sigma_{12}\rangle}{\sigma_{y.ef}}\right)^2\right] = 1 \, . \tag{8}$$

The effective yield stress $\sigma_{y.ef}$ can be determined from Eq. (8) as a function of the properties of the components $\sigma_{y.i}$, $i = 1, 2$.

We will determine the effective yield stress $\sigma_{y.ef}$ by the method of elastic solutions [5], which is based on successive approximations which allow one to replace calculations of an elastico-plastic plate by calculations of an elastic plate subjected to a uniformly distributed load whose value is refined in the process of iterations.

For the majority of metals the Poisson coefficient ν for an elastic region is assumed to be equal to 1/3. On transition to the region of plastic deformations, it increases and attains a value of 1/2. At this value of ν the relative change in the volume of the body as a result of plastic deformations is equal to zero, i.e., the material behaves as an incompressible one [4, 6]. Therefore, for the region of plasticity in a finite-difference scheme of the solution of the elasticity problem by the method of integrated cross-sections [7] it is necessary to replace the flexural rigidity in Eq. (2) by $D = Eh^3/9$, i.e., by the flexural rigidity at $\nu = 1/2$.

The calculation of an elastic plate is reduced to the solution of a system of linear algebraic equations:

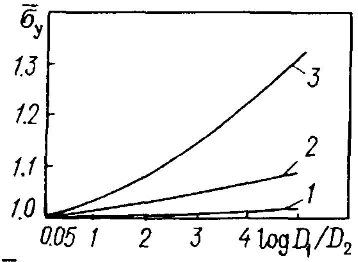

Fig. 1. Dependence of $\bar{\sigma}_y = \sigma_{y.ef}/\sigma_y$ on the D_1/D_2 logarithm: 1) $v = 22.2\%$, 2) 44.4, 3) 55.6.

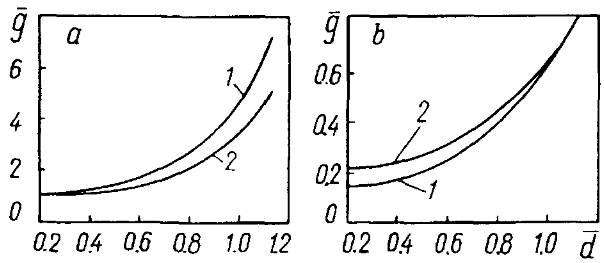

Fig. 2. Dependence of $\bar{d} = d_{pl}/a$ on $\bar{g} = g/g_1^0$ (a) and g/g_2^0 (b): a, 1) inhomogeneous plate, $D < D_1$, 2) homogeneous plate, $D = D_1$; b, 1) inhomogeneous plate, $D_1 < D_2$, 2) homogeneous plate, $D = D_2$.

$$L\mathbf{w} = \mathbf{g}\,, \tag{9}$$

where L is the band matrix of the coefficients in the system of linear algebraic equations obtained from Eqs. (1)-(3) using the finite-difference scheme of the method of integrated cross-sections for the elasticity problem.

We find the unknown saggings of the plate by solving system (9); to do this, it is sufficient to calculate the inverse matrix L^{-1} of the coefficients in system (9) and multiply it by the vector g:

$$\mathbf{w} = L^{-1}\mathbf{g}\,.$$

Thus, using the method of elastic solutions, the nonlinear problem is reduced to the solution of a sequence of systems of linear algebraic equations of the form

$$L\mathbf{w}_n = \mathbf{g}_n\,,$$

where $\mathbf{g}_n = \mathbf{g} + \mathbf{g}_{n-1}$, $n > 0$.

At each step of the iteration process we verify the Huber–Mises condition (7) for each of the components into which the inhomogeneous plate is divided according to the method of integrated cross-sections. The iteration process continues until the difference between the two last approximations is considered to be small.

Results of Calculations. Figures 1 and 2 present the results of calculations for a plate of thickness $h = 0.01$, $a = 1.8$.

On the basis of numerical data (Fig. 1) we may conclude that when the inclusion is "stiffer" that the matrix, the growth of its geometrical parameters does not exert a substantial influence on the effective yield stress of the inhomogeneous plate until the concentration of the inclusion exceeds 23%.

The graphs given in Fig. 2 are plotted for $D_2/D_1 = 1.5$ and $v = 55.6\%$. Analysis of the results presented in Fig. 2a allows the following conclusion: in an inhomogeneous plate the stiffness of whose matrix is an order of magnitude larger than the stiffness of the inclusion and in a homogeneous plate whose stiffness is equal to that of the inclusion of an inhomogeneous plate, the plasticity zone appears at the same loading. With a further increase in the latter, the plasticity zone propagates identically until it occupies 50% of the area of the corresponding plates. For this to occur, the loading must increase 1.5-fold compared to that at which the plasticity zone appeared. With a further increase in loading, the plates behave differentlty: an inhomogeneous plate reinforced by a "stiff" matrix changes into a plastic state completely at a loading that is 1.3 times larger than for a homogeneous plate.

A quantitative analysis of the data, presented in Fig. 2b, allows us to conclude that the plasticity zone in inhomogeneous and homogeneous plates (the stiffness of the latter is equal to that of the matrix of the inhomogeneous plate) appears at a different loading, namely, in an inhomogeneous plate weakened by a "soft" inclusion, the plasticity zone appears at a loading that is 1.6 times smaller than in the given homogeneous plate.

The values of the limiting loadings at which the plates considered change completely into the plastic state correspond to the upper estimate of the limiting loading obtained in [3].

Conclusion. We solved the elastico-plastic problem of the bending of a Kirchhoff inhomogeneous square plate. The dependences of its effective yield stress on the geometrical and stiffness characteristics of the matrix and inclusion are obtained. We carried out a comparative analysis of the elastico-plastic properties of inhomogeneous and homogeneous plates. The results of calculations on an electronic computer agree well with formula (5.38) from [3], p. 116. The method of integrated cross-sections together with the method of elastic solutions can be conveniently used in engineering practice.

NOTATION

E, Young modulus; v, concentration of inclusion; a, length of square plate side; d_{pl}, diameter of plastic zone; g_1^0, loading under which plasticity zone appears in plate; g_2^0, loading under which plate changes completely to plastic state. Subscripts: y, yield; y1, yield of material 1; y2, yield of the material 2; y.ef., effective yield.

REFERENCES

1. V. A. Rvachev and N. S. Sinekop, The Method of R-functions in Problems of the Theory of Elasticity and Plasticity [in Russian], Kiev (1990).
2. V. D. Klyushnikov, Mathematical Theory of Plasticity [in Russian], Moscow (1979).
3. A. A. Il'yushin, Plasticity [in Russian], Moscow (1948).
4. S. I. Terebushko, Fundamentals of the Theory of Elasticity and Plasticity [in Russian], Moscow (1948).
5. A. V. Aleksandrov and V. D. Potapov, Fundamentals of the Theory of Elasticity and Plasticity: Textbook [in Russian], Moscow (1990).
6. V. Prager and F. G. Hodge, Theory of Perfectly Plastic Bodies [Russian translation], Moscow (1956).
7. V. P. Privalko, V. V. Novikov, and Yu. F. Yanovskii, Fundamentals of Thermal Physics and Rheophysics of Polymer Materials [in Russian], Kiev (1991).